

$$\int \frac{dx}{x^3+1}, \int \frac{dx}{x^4+1}, \int \sqrt{\tan \theta} d\theta, \int \sqrt{\cot \theta} d\theta$$

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$$\begin{aligned} 1. \int \frac{dx}{x^3+1} &= \int \frac{dx}{(x+1)(x^2-x+1)} \\ &= \frac{1}{3} \int \left(\frac{1}{x+1} - \frac{x-2}{x^2-x+1} \right) dx && \text{(Break integrand by Partial Fraction)} \\ &= \frac{1}{3} \int \frac{dx}{x+1} - \frac{1}{6} \int \frac{(2x-1)-3}{x^2-x+1} dx \\ &= \frac{1}{3} \int \frac{dx}{x+1} - \frac{1}{6} \int \frac{(2x-1)-3}{x^2-x+1} dx \\ &= \frac{1}{3} \int \frac{dx}{x+1} - \frac{1}{6} \int \frac{2x-1}{x^2-x+1} dx + \frac{1}{2} \int \frac{1}{x^2-x+1} dx \\ &= \frac{1}{3} \int \frac{d(x+1)}{x+1} - \frac{1}{6} \int \frac{d(x^2-x+1)}{x^2-x+1} + \frac{1}{2} \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx \\ &= \frac{1}{3} \ln(x+1) - \frac{1}{6} \ln(x^2-x+1) + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right) + C \end{aligned}$$

$$\text{For } I = \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx, \quad \text{let } x - \frac{1}{2} = \frac{\sqrt{3}}{4} \tan \theta, \quad dx = \frac{\sqrt{3}}{4} \sec^2 \theta d\theta$$

$$\begin{aligned} \therefore I &= \int \frac{\frac{\sqrt{3}}{4} \sec^2 \theta d\theta}{\frac{3}{4} \sec^2 \theta} = \frac{2}{\sqrt{3}} \int d\theta \\ &= \frac{2}{\sqrt{3}} \theta + C = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right) + C \end{aligned}$$

$$2. \int \frac{dx}{x^4+1}$$

Method 1

$$I = \int \frac{dx}{x^4+1} = \int \frac{dx}{(x^2+1)^2 - 2x^2} = \int \frac{dx}{(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)}$$

The partial fraction is clumsy:

$$\frac{1}{(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)} = \frac{Ax + B}{x^2 + \sqrt{2}x + 1} + \frac{Cx + D}{x^2 - \sqrt{2}x + 1}$$

We get:

$$(Ax + B)(x^2 - \sqrt{2}x + 1) + (Cx + D)(x^2 + \sqrt{2}x + 1) = 1$$

We need to solve for A, B, C, D with 4 equations involving surds.

To avoid this, please study the following:

$$(a) (x^2 + \sqrt{2}x)(x^2 - \sqrt{2}x) - (x^2 - \sqrt{2}x)(x^2 + \sqrt{2}x) = 0$$

$$x(x + \sqrt{2})(x^2 - \sqrt{2}x) - x(x - \sqrt{2})(x^2 + \sqrt{2}x) = 0$$

Cancel x in (a), we get:

$$(b) (x + \sqrt{2})(x^2 - \sqrt{2}x) - (x - \sqrt{2})(x^2 + \sqrt{2}x) = 0$$

$$(c) (x + \sqrt{2})1 - (x - \sqrt{2})1 = 2\sqrt{2}$$

Adding (b) and (c), we get:

$$(d) (x + \sqrt{2})(x^2 - \sqrt{2}x + 1) - (x - \sqrt{2})(x^2 + \sqrt{2}x + 1) = 2\sqrt{2}$$

$$(e) \frac{1}{(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)} = \frac{1}{2\sqrt{2}} \left(\frac{x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} - \frac{x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} \right)$$

$$\begin{aligned} \therefore I &= \frac{1}{2\sqrt{2}} \int \frac{x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} dx - \frac{1}{2\sqrt{2}} \int \frac{x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} dx \\ &= \frac{1}{4\sqrt{2}} \int \frac{d(x^2 + \sqrt{2}x + 1)}{x^2 + \sqrt{2}x + 1} + \frac{1}{4} \int \frac{dx}{x^2 + \sqrt{2}x + 1} - \frac{1}{4\sqrt{2}} \int \frac{d(x^2 - \sqrt{2}x + 1)}{x^2 - \sqrt{2}x + 1} + \frac{1}{4} \int \frac{dx}{x^2 - \sqrt{2}x + 1} \\ &= \frac{1}{4\sqrt{2}} \ln \left(\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right) + \frac{1}{4} \int \frac{dx}{\left(x + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} + \frac{1}{4} \int \frac{dx}{\left(x - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \\ &= \frac{\sqrt{2}}{8} \ln \left(\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right) + \frac{\sqrt{2}}{4} \tan^{-1}(\sqrt{2}x + 1) + \frac{\sqrt{2}}{4} \tan^{-1}(\sqrt{2}x - 1) + C \\ &= \frac{\sqrt{2}}{8} \ln \left(\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right) + \frac{\sqrt{2}}{4} [\tan^{-1}(\sqrt{2}x + 1) + \tan^{-1}(\sqrt{2}x - 1)] + C \\ &= \frac{\sqrt{2}}{8} \ln \left(\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right) + \frac{\sqrt{2}}{4} \left[\tan^{-1} \frac{(\sqrt{2}x + 1) + (\sqrt{2}x - 1)}{1 - (\sqrt{2}x + 1)(\sqrt{2}x - 1)} \right] + C \\ &= \frac{\sqrt{2}}{8} \ln \left(\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right) + \frac{\sqrt{2}}{4} \tan^{-1} \frac{\sqrt{2}x}{1 - x^2} + C \quad \dots \quad (1) \end{aligned}$$

Method 2

$$\begin{aligned} I &= \int \frac{dx}{x^4 + 1} = \frac{1}{2} \int \frac{(x^2 + 1) - (x^2 - 1)dx}{x^4 + 1} \\ &= \frac{1}{2} \int \frac{x^2 + 1}{x^4 + 1} dx - \frac{1}{2} \int \frac{x^2 - 1}{x^4 + 1} dx \\ &= \frac{1}{2} I_1 - \frac{1}{2} I_2 \end{aligned}$$

$$\text{For } I_1 = \int \frac{x^2+1}{x^4+1} dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx$$

$$\text{Let } u = x - \frac{1}{x}, \quad du = \left(1 + \frac{1}{x^2}\right) dx$$

$$\text{And } x^2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2 + 2 = u^2 + 2$$

$$\begin{aligned} \therefore I_1 &= \int \frac{du}{u^2 + 2} = \frac{\sqrt{2}}{2} \tan^{-1} \frac{u}{\sqrt{2}} + C_1 \\ &= \frac{\sqrt{2}}{2} \tan^{-1} \frac{1}{\sqrt{2}} \left(x - \frac{1}{x}\right) + C_1 = \frac{\sqrt{2}}{2} \tan^{-1} \frac{x^2-1}{\sqrt{2}x} + C_1 \end{aligned}$$

$$\text{For } I_2 = \int \frac{x^2-1}{x^4+1} dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx$$

$$\text{Let } u = x + \frac{1}{x}, \quad du = \left(1 - \frac{1}{x^2}\right) dx$$

$$\text{And } x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = u^2 - 2$$

$$\begin{aligned} \therefore I_2 &= \int \frac{du}{u^2 - 2} = \int \frac{du}{(u - \sqrt{2})(u + \sqrt{2})} \\ &= \frac{1}{2\sqrt{2}} \left(\int \frac{du}{u - \sqrt{2}} - \int \frac{du}{u + \sqrt{2}} \right) \\ &= \frac{\sqrt{2}}{4} \ln \left(\frac{u - \sqrt{2}}{u + \sqrt{2}} \right) + C_2 \\ &= \frac{\sqrt{2}}{4} \ln \left(\frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right) + C_2 \\ &= \frac{\sqrt{2}}{4} \ln \left(\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right) + C_2 \end{aligned}$$

$$\begin{aligned} \therefore I &= \frac{1}{2} I_1 - \frac{1}{2} I_2 = \frac{\sqrt{2}}{4} \tan^{-1} \frac{x^2 - 1}{\sqrt{2}x} - \frac{\sqrt{2}}{8} \ln \left(\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right) + C \\ &= \frac{\sqrt{2}}{4} \tan^{-1} \frac{x^2-1}{\sqrt{2}x} + \frac{\sqrt{2}}{8} \ln \left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} \right) + C \quad \dots (2) \end{aligned}$$

Method 3 (Best)

$$\begin{aligned} I &= \int \frac{dx}{x^4+1} = \frac{1}{2} \int \frac{(x^2+1)-(x^2-1)dx}{x^4+1} = \frac{1}{2} \int \frac{x^2+1}{x^4+1} dx - \frac{1}{2} \int \frac{x^2-1}{x^4+1} dx = \frac{1}{2} \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx - \frac{1}{2} \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx \\ &= \frac{1}{2} \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2} dx - \frac{1}{2} \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2} dx = \frac{1}{2\sqrt{2}} \tan^{-1} \frac{1}{\sqrt{2}} \left(x - \frac{1}{x}\right) - \frac{1}{4\sqrt{2}} \ln \frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}} + C \quad \dots (3) \end{aligned}$$

Method 4 (Using Complex numbers)

$$\frac{1}{x^4+1} = \frac{1}{(x^2-i)(x^2+i)} = \frac{1}{2i} \left(\frac{1}{x^2-i} - \frac{1}{x^2+i} \right) = \frac{1}{2i} \left(\frac{1}{x^2-(\sqrt{i})^2} - \frac{1}{x^2+(\sqrt{i})^2} \right)$$

$$I = \int \frac{dx}{x^4+1} = \frac{1}{2i} \left(\int \frac{dx}{x^2-(\sqrt{i})^2} - \int \frac{dx}{x^2+(\sqrt{i})^2} \right) = \frac{1}{4i\sqrt{i}} \ln \left(\frac{x-\sqrt{i}}{x+\sqrt{i}} \right) - \frac{1}{2i\sqrt{i}} \tan^{-1} \left(\frac{x}{\sqrt{i}} \right) + C \quad \dots (4)$$

The method is simple, but the appearance of many complex numbers seems uncomfortable.

3. The result of Method 1 and Method 2 above differ by a constant.

$$\tan^{-1} \frac{x^2-1}{\sqrt{2x}} - \tan^{-1} \frac{\sqrt{2x}}{1-x^2} = \tan^{-1} \frac{\frac{x^2-1}{\sqrt{2x}} - \frac{\sqrt{2x}}{1-x^2}}{1 + \left(\frac{x^2-1}{\sqrt{2x}} \right) \left(\frac{\sqrt{2x}}{1-x^2} \right)} = \tan^{-1} \infty = \frac{\pi}{2}$$

4. $\int \sqrt{\tan \theta} d\theta, \int \sqrt{\cot \theta} d\theta$

$$\int \frac{dx}{x^4+1} = \frac{\sqrt{2}}{4} \tan^{-1} \frac{x^2-1}{\sqrt{2x}} + \frac{\sqrt{2}}{8} \ln \left(\frac{x^2 + \sqrt{2x} + 1}{x^2 - \sqrt{2x} + 1} \right) + C$$

(a) Put $x = \sqrt{\tan \theta}, \quad dx = \frac{\sec^2 \theta}{2\sqrt{\tan \theta}} d\theta$

$$\int \frac{dx}{x^4+1} = \int \frac{\sec^2 \theta}{(\tan^2 \theta + 1)2\sqrt{\tan \theta}} d\theta = \frac{1}{2} \int \sqrt{\cot \theta} d\theta$$

$$\therefore \int \sqrt{\cot \theta} d\theta = \frac{\sqrt{2}}{2} \tan^{-1} \frac{\tan \theta - 1}{\sqrt{2 \tan \theta}} + \frac{\sqrt{2}}{4} \ln \left(\frac{\tan \theta + \sqrt{2 \tan \theta} + 1}{\tan \theta - \sqrt{2 \tan \theta} + 1} \right) + C$$

(b) Put $x = \sqrt{\cot \theta}, \quad dx = \frac{-\csc^2 \theta}{2\sqrt{\cot \theta}} d\theta$

$$\int \frac{dx}{x^4+1} = \int \frac{-\csc^2 \theta}{(\cot^2 \theta + 1)2\sqrt{\cot \theta}} d\theta = -\frac{1}{2} \int \sqrt{\tan \theta} d\theta$$

$$\int \sqrt{\tan \theta} d\theta = -\frac{\sqrt{2}}{2} \tan^{-1} \frac{\cot \theta - 1}{\sqrt{2 \cot \theta}} - \frac{\sqrt{2}}{4} \ln \left(\frac{\cot \theta + \sqrt{2 \cot \theta} + 1}{\cot \theta - \sqrt{2 \cot \theta} + 1} \right) + C$$