Volume of Frustum

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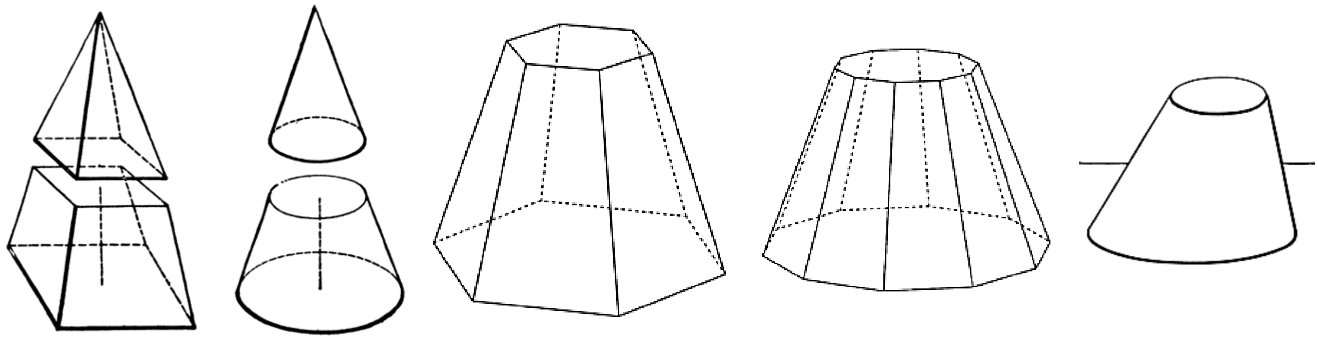

A frustum is a truncated cone or pyramid; the part that is left when a cone or pyramid is cut by a plane parallel to the base with the apical part removed. We concentrate on the right frustum and not slant frustum as in the right-most diagram on the top.

We begin with :

Volume of prism $= \frac{1}{3}Bh$, where B is the base area and h is the height of prism.

Since the apical part is removed, we have:

Volume of frustum, V $= \frac{1}{3}\left(B_2 h_2 - B_1 h_1\right)$ …. (1)

The height of the frustum $= h = h_2 - h_1$ …. (2)

We also have : $\frac{B_2}{B_1} = \frac{{h_2}^2}{{h_1}^2}$ …. (3)

Our aim is to remove the variables h_1, h_2 using (1) – (3) .

From (2), $h_2 = h + h_1$ …. (4)

(4) ↓ (1) , $V = \frac{1}{3}\left[B_2(h + h_1) - B_1 h_1\right] = \frac{1}{3}\left[B_2 h + (B_2 h_1 - B_1 h_1)\right]$

$= \frac{1}{3}\left[B_2 h + (B_2 - B_1)h_1\right]$ …. (5)

From (3), ${h_2}^2 = \frac{{h_1}^2 B_2}{B_1} \Rightarrow h_2 = h_1 \frac{\sqrt{B_2}}{\sqrt{B_1}}$ …. (6)

(6) ↓ (2) , $h = h_1 \frac{\sqrt{B_2}}{\sqrt{B_1}} - h_1 = \frac{\sqrt{B_2} - \sqrt{B_1}}{\sqrt{B_1}} h_1 \Rightarrow h_1 = \frac{\sqrt{B_1}}{\sqrt{B_2} - \sqrt{B_1}} h$ …. (7)

(7) ↓ (5) , $$V = \frac{1}{3}\left[B_2h + (B_2 - B_1)\frac{\sqrt{B_1}}{\sqrt{B_2} - \sqrt{B_1}}h\right]$$

$$= \frac{1}{3}\left[B_2h + \left(\sqrt{B_2} + \sqrt{B_1}\right)\left(\sqrt{B_2} - \sqrt{B_1}\right)\frac{\sqrt{B_1}}{\sqrt{B_2} - \sqrt{B_1}}h\right]$$

$$= \frac{1}{3}\left[B_2h + \sqrt{B_1}\left(\sqrt{B_2} + \sqrt{B_1}\right)h\right]$$

$$= \frac{h}{3}\left[B_1 + \sqrt{B_1B_2} + B_2\right] \qquad \text{…. (8)}$$

From (8), we can derive the followings:

(a) Volume of circular cone frustum,

$$V = \frac{\pi h}{3}\left[r_1^2 + r_1r_2 + r_2^2\right]$$

, where r_1, r_2 are the radii of the top and bottom circular bases.

(b) Volume of square frustum,

$$V = \frac{h}{3}\left[a_1^2 + a_1a_2 + a_2^2\right]$$

, where a_1, a_2 are the sides of the top and bottom square bases.

(c) Since the area of equilateral triangle is given by

$A = \frac{\sqrt{3}}{4}a^2$, where a is the side of the triangle, we have:

Volume of equilateral triangular frustum,

$$V = \frac{h}{16}\left[a_1^2 + a_1a_2 + a_2^2\right]$$

, where a_1, a_2 are the sides of the top and bottom triangular bases.