Equation of sphere through four points

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(1) Email from S.S. (name initialized for confidentiality) dated 22-2-2016

> Hi,
>
> http://www.qc.edu.hk/math/Advanced%20Level/circle%20given%203%20points.htm
>
> I found your webpage at the link above very interesting. I am having a hard time figuring out how to modify the "System of Circles" method to work in 3D for spheres. I asked the question over on the website below. And I thought I would send you the link in case you could offer any insight.
>
> http://math.stackexchange.com/questions/1666927/how-to-adapt-system-of-circles-method-to-3d-for-finding-a-sphere-given-4-point
>
> Thanks!

(2) My reply email on 25-2-2016

Let me illustrate my method by finding the sphere passing through the points:

$$A(1,-1,3), B(4,1,-2), C(-1,-1,1), D(1,1,1)$$

It is noted that in order to have a unique sphere solution, any three of the four given points must not be co-linear and all four points must not be co-planar.

(A) First we like to find the equation of plane P passing through the first three points :

$$A(1,-1,3), B(4,1,-2), C(-1,-1,1)$$

$$\overrightarrow{AB} = (3,2,-5), \overrightarrow{BC} = (-5,-2,3)$$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} i & j & k \\ 3 & 2 & -5 \\ -5 & -2 & 3 \end{vmatrix} = (-4,16,4)$$

Let $\vec{X} = (x, y, z)$ be any point on the plane.

Equation of plane P is therefore given by $\overrightarrow{XA} \cdot \vec{n} = 0$

$$-4(x-1) + 16(y+1) + 4(z-3) = 0$$

$$4x - 16y - 4z = 8$$

$$x - 4y - z = 2 \quad \ldots.(1)$$

(B) We like to find the centre of the circle passing through the first three points :

$$A(1,-1,3), B(4,1,-2), C(-1,-1,1)$$

Let $E(x,y,z)$ be the required centre.

Then $|EA| = |EB| = |EC| = r^2$, where r is the radius of the circle. Therefore

$$(x-1)^2 + (y+1)^2 + (z-3)^2 = (x-4)^2 + (y-1)^2 + (z+2)^2 = (x+1)^2 + (y+1)^2 + (z-1)^2$$

(i) $(x-1)^2 + (y+1)^2 + (z-3)^2 = (x-4)^2 + (y-1)^2 + (z+2)^2$

$$6x + 4y - 10z = 10$$

$$3x + 2y - 5z = 5 \quad \ldots.(2)$$

(ii) $(x-4)^2 + (y-1)^2 + (z+2)^2 = (x+1)^2 + (y+1)^2 + (z-1)^2$

$$10x + 4y - 6z = 18$$

$$5x + 2y - 3z = 9 \quad \ldots.(3)$$

Note that $(x+1)^2 + (y+1)^2 + (z-1)^2 = (x-1)^2 + (y+1)^2 + (z-3)^2$ will give an equation dependent on **(i)** and **(ii)** , and is therefore a redundant equation.

Solving **(i)** and **(ii)** will give the normal straight line passing through the centre E.

This normal passes through the plane got in **(A)** will give the centre of the required circle.

We do not like to solve **(i)** and **(ii)** here, instead we solve **(A)**, **(i)** and **(ii)** altogether.

$$\begin{cases} x - 4y - z = 2 & \ldots.(1) \\ 3x + 2y - 5z = 5 & \ldots.(2) \\ 5x + 2y - 3z = 9 & \ldots.(3) \end{cases}$$

We get $x = \frac{17}{9}, y = -\frac{1}{18}, z = \frac{1}{9}$. $\quad \therefore E = \left(\frac{17}{9}, -\frac{1}{18}, \frac{1}{9}\right)$

Also radius of the circle is r where $\quad r^2 = \left(\frac{17}{9} - 1\right)^2 + \left(-\frac{1}{18} + 1\right)^2 + \left(\frac{1}{9} - 3\right)^2 = \frac{361}{36}$

(C) We then find a sphere S with centre E and radius r.

Note that is sphere has the same centre and radius as the circle in part **(B)**.

As a result, the circle in **(B)** is the equatorial great circle of the sphere S.

Equation of a sphere through $A(1,-1,3), B(4,1,-2), C(-1,-1,1)$

and centre $E = \left(\frac{17}{9}, -\frac{1}{18}, \frac{1}{9}\right)$ is simply got as follows:

If (x,y,z) is a variable point on this sphere, then the distance from $A(1,-1,3)$ is the radius r.

Therefore S: $\left(x - \frac{17}{9}\right)^2 + \left(y + \frac{1}{18}\right)^2 + \left(z - \frac{1}{9}\right)^2 = \frac{361}{36} \quad \ldots.(4)$

(D) Finally, we form a ***system of spheres***: $S + kP = 0$

$$\left(x-\frac{17}{9}\right)^2+\left(y+\frac{1}{18}\right)^2+\left(z-\frac{1}{9}\right)^2-\frac{361}{36}+k(x-4y-\ z-2)=0 \quad \dots.(5)$$

The last point $D(1,1,1)$ satisfies this equation. On substitution we get

$$\left(1-\frac{17}{9}\right)^2+\left(1+\frac{1}{18}\right)^2+\left(1-\frac{1}{9}\right)^2-\frac{361}{36}+k(1-4-\ 1-2)=0$$

$$\therefore\ k=-\frac{11}{9}$$

The sphere passing through all four points $A(1,-1,3), B(4,1,-2), C(-1,-1,1), D(1,1,1)$ is

$$\left(x-\frac{17}{9}\right)^2+\left(y+\frac{1}{18}\right)^2+\left(z-\frac{1}{9}\right)^2-\frac{361}{36}-\frac{11}{9}(x-4y-\ z-2)=0$$

$$\left(x-\frac{17}{9}\right)^2+\left(y+\frac{1}{18}\right)^2+\left(z-\frac{1}{9}\right)^2-\frac{361}{36}-\frac{11}{9}(x-4y-\ z-2)=0$$

$$x^2+y^2+z^2-5\,x+5\,y+z-4=0 \quad \dots.(6)$$

(E) **Another method**

In **(B)(i)** and **(ii)** , we found two equations in which the intersections gives us the normal line:

$$\begin{cases}3\,x+2\,y-5\,z=5 & \dots.(2)\\ 5\,x+2\,y-3\,z=9 & \dots.(3)\end{cases}$$

In fact, we can choose **any** convenient point in the normal, for example:

Put $z=0$, $\begin{cases}3\,x+2\,y=5 & \dots.(7)\\ 5\,x+2\,y=9 & \dots.(8)\end{cases}$

$(8)-(7)$, $\quad 2\,x=4 \Longrightarrow x=2$ and $y=\ -\frac{1}{2}$

And therefore $F=\left(2,-\frac{1}{2},0\right)$ is a point on the normal.

Use $r_1=|FA|=|FB|=|FB|$ as radius, we get

$$r_1{}^2=(2-1)^2+\left(-\frac{1}{2}+1\right)^2+(0-3)^2=\frac{41}{4}$$

We can form a sphere :

$$S_1:\quad (x-2)^2+\left(y+\frac{1}{2}\right)^2+(z-0)^2=\frac{41}{4} \quad \dots.(7)$$

This sphere is of course different from the sphere S got in equation (4).

However, $A(1,-1,3), B(4,1,-2), C(-1,-1,1)$ is on S_1.

Form a system of spheres:

$$S_1+k_1P:\ (x-2)^2+\left(y+\frac{1}{2}\right)^2+(z-0)^2-\frac{41}{4}+k_1(x-4y-\ z-2)=0$$

Substitute point $D(1,1,1)$,

$$(1-2)^2+\left(1+\frac{1}{2}\right)^2+(1-0)^2-\frac{41}{4}+k_1(1-4-\ 1-2)=0$$

$$\therefore k_1=-1$$

And the equation of the sphere passing through all four points is

$$(x-2)^2+\left(y+\frac{1}{2}\right)^2+(z-0)^2-\frac{41}{4}-(x-4y-\ z-2)=0$$

$$x^2+y^2+z^2-5\,x+5\,y+z-4=0 \quad \dots.(6)$$

(F) Determinant method

The equation of a sphere passing through four points is given by Beyer 1987:

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix}=0$$

The determinant $\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 1^2+(-1)^2+3^2 & 1 & -1 & 3 & 1 \\ 4^2+1^2+(-2)^2 & 4 & 1 & -2 & 1 \\ (-1)^2+(-1)^2+1^2 & -1 & -1 & 1 & 1 \\ 1^2+1^2+1^2 & 1 & 1 & 1 & 1 \end{vmatrix}$

$$=-24x^2-24y^2-24z^2+120x-120y-24z+96$$

This determinant needs some time to evaluate, instead I put it in Wolfram to get the result.

Therefore the equation of the sphere is

$$-24x^2-24y^2-24z^2+120x-120y-24z+96=0$$

Or $\quad x^2+y^2+z^2-5x+5y+z-4=0 \quad \dots.(6)$