An Envelope Problem

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A curve or a surface which touches every member of a family of lines, curves, planes or surfaces. In this article, we discuss an envelope problem involving a family of straight lines.

Consider a family of straight line:

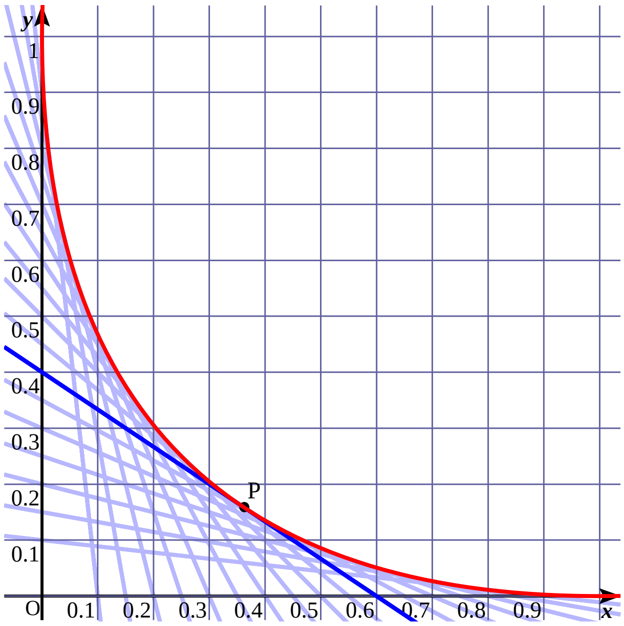
$$C_t: \frac{x}{t} + \frac{y}{1-t} = 1, \qquad \text{where } t \in [0,1]$$

The diagram on the right shows the family of straight lines in blue colour.

One of these straight lines for which t = 0.6 is shown in dark blue.

The envelope of the family of straight lines is shown in red. It touches all straight lines and as an example, the straight line with t = 0.6 at the point P.

We are going to find the **equation** of the envelope.



Now,

$$\frac{x}{t} + \frac{y}{1-t} = 1 \qquad \text{.... (1)}$$

$$x(1-t) + yt = t(1-t)$$

$$t^2 - (x - y + 1)t + x = 0 \qquad \text{.... (2)}$$

(2) is a quadratic equation in t and since the envelope touches the family of straight line at only **one** point,

$$\Delta = [-(x-y+1)]^2 - 4(1)(x) = 0$$

On expanding, we get $x^2 + y^2 - 2xy - 2x - 2y + 1 = 0$ .... (3)

(3) is an homogeneous equation in x and y and is of degree 2. It therefore represents a **conic** curve. The shape of the curve in the above diagram looks like a parabola or a branch of a hyperbola. We proceed to investigate which is correct?

We rotate the axes x-y anti-clockwisely by an angle θ to a new axes X-Y using :

$$\begin{cases} x = X\cos\theta - Y\sin\theta \\ y = X\sin\theta + Y\cos\theta \end{cases} \quad .... \quad (4)$$

Equation (3) becomes:

$$(X\cos\theta - Y\sin\theta)^2 + (X\sin\theta + Y\cos\theta)^2 - 2(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) - 2(X\cos\theta - Y\sin\theta) - 2(X\sin\theta + Y\cos\theta) + 1 = 0 \quad .... \quad (5)$$

If the coefficient of XY term = 0, then

$$-2(\cos^2\theta - \sin^2\theta) = 0$$

$$\therefore \cos 2\theta = 0$$

For simplicity, we take one of the root: $\theta = \frac{\pi}{4}$.

(4) becomes :

$$\begin{cases} x = X\cos\frac{\pi}{4} - Y\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(X - Y) \\ y = X\sin\frac{\pi}{4} + Y\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}(X + Y) \end{cases} \quad .... \quad (6)$$

Subst. (6) in (5) :

$$\left(\frac{1}{\sqrt{2}}(X - Y)\right)^2 + \left(\frac{1}{\sqrt{2}}(X + Y)\right)^2 - 2\left(\frac{1}{\sqrt{2}}(X - Y)\right)\left(\frac{1}{\sqrt{2}}(X + Y)\right) - 2\left(\frac{1}{\sqrt{2}}(X - Y)\right) - 2\left(\frac{1}{\sqrt{2}}(X + Y)\right) + 1 = 0$$

or $\frac{X^2+2XY+Y^2}{2} + \frac{X^2-2XY+Y^2}{2} - (X^2 - Y^2) - \sqrt{2}(X - Y) - \sqrt{2}(X - Y) + 1 = 0$

or $2Y^2 - 2\sqrt{2}X + 1 = 0$

or $Y^2 = \sqrt{2}X - \frac{1}{2}$

or $Y^2 = \sqrt{2}\left(X - \frac{\sqrt{2}}{4}\right)$

A further simple translation of axes (not shown here) will give a **parabola** in standard form.

Can we find the Area enclosed by this envelope in the first quadrant?

Let us start with earlier equation of envelope :

$$x^2 + y^2 - 2xy - 2x - 2y + 1 = 0 \qquad .... \quad (3)$$

We need to change this to explicit function form. Combining terms in (3) we get:

$$y^2 - 2y(x+1) + (x^2 - 2x + 1) = 0$$
$$y^2 - 2y(x+1) + (x-1)^2 = 0$$

which is a quadratic equation in y.

By quadratic equation formula,

$$y = \frac{2(x+1) \pm \sqrt{4(x+1)^2 - 4(x-1)^2}}{2} = (x+1) \pm 2\sqrt{x} \qquad .... \quad (7)$$

Since $y \leq 1$, we choose the negative root ,

$$y = x + 1 - 2\sqrt{x} \qquad .... \quad (8)$$

which is the equation of envelope in explicit form.

The area enclosed by (5) with the x and y axes is now easy :

$$\int_0^1 (x + 1 - 2\sqrt{x})dx = \left[\frac{x^2}{2} + x - \frac{4}{3}x^{3/2}\right]_0^1 = \underline{\underline{\frac{1}{6}}}$$