An example on five classical centres of a right angled triangle

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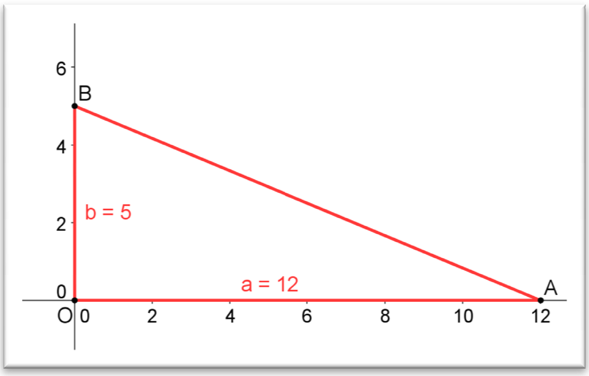
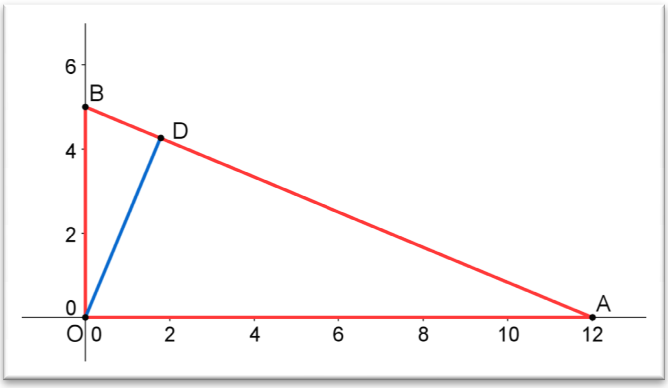
Given O(0, 0), A(12, 0), B(0, 5) .

The aim of this small article is to find the co-ordinates of the ***five classical centres*** of the Δ OAB and other related points of interest. We choose a right-angled triangle for simplicity.



(1) Orthocenter:

The three ***altitudes*** of a triangle meet in one point called the orthocenter. (Altitudes are perpendicular lines from vertices to the opposite sides of the triangles.)



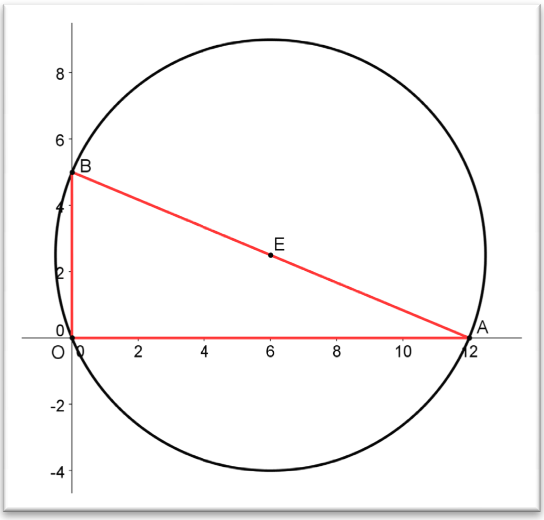
If the triangle is obtuse, the orthocenter is outside the triangle. If it is a right triangle, the orthocenter is the vertex which is the right angle.

Conclusion : Simple, the orthocenter = O(0, 0) .

(2) Circum-center:

The three ***perpendicular bisectors*** of the sides of a triangle meet in one point called the circumcenter. It is the center of the circumcircle, the circle circumscribed about the triangle.

If the triangle is obtuse, then the circumcenter is outside the triangle. If it is a right triangle, then the circumcenter is the midpoint of the hypotenuse. (By the theorem of angle in semi-circle as in the diagram.)



Conclusion : the circum-centre, $E = \left(\frac{12+0}{2}, \frac{0+5}{2}\right) = (6,\ \ 2.5)$

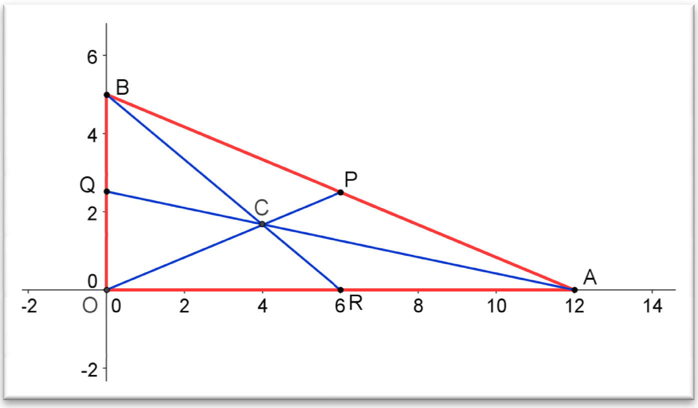
Exercise 1:

(a) Check that the circum-circle above is given by: $(x-6)^2+(y-2.5)^2=6.5^2$.

(b) Show that the area of the triangle with sides a, b, c and angles A, B, C is

$$\frac{abc}{4R} = 2R^2 \sin A \sin B \sin C \ , \qquad \text{where R is the radius of the circum-circle .}$$

(3) Centroid:



The three ***medians*** (the lines drawn from the vertices to the bisectors of the opposite sides) meet in the centroid or center of mass. The centroid divides each median in a ratio of 2 : 1.

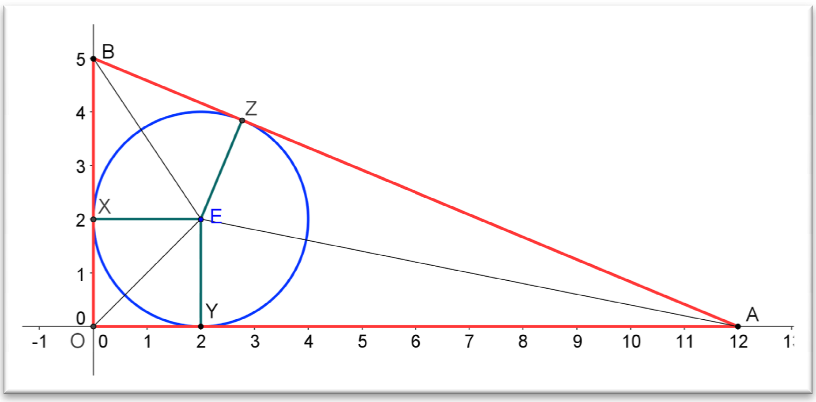
Since OR : RA = 1 : 1, we have $R = (6, 0)$.

Since BC : CR = 2 : 1, we therefore have

Conclusion : Centroid, $C = \left(\frac{2\times6+1\times0}{2+1}, \frac{2\times0+1\times2.5}{2+1}\right) = \left(4, \frac{5}{3}\right)$.

Exercise 2: Prove that if $A=(x_1,y_1), B=(x_2,y_2), C=(x_3,y_3)$, then the coordinates of the centroid of ΔABC is given by $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

(4) In-center:



In-center: The three ***angle bisectors*** of a triangle meet in one point called the in-center. It is the center of the in-circle, the circle inscribed in the triangle.

Let OA = a = 12,

OB = b = 5

AB = c = $\sqrt{12^2+5^5}$ =13

Semi-perimeter, $s = \frac{a+b+c}{2} = \frac{12+5+13}{2} = 15$

The radius of the incircle = r = EX = EY = EZ

Then Area of ΔOAB = Area of ΔEOA + Area of ΔEOB+ Area of ΔEAB

$$\frac{ab}{2} = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} = \left(\frac{a+b+c}{2}\right) r = sr$$

$$\therefore \quad r = \frac{ab}{2s} = \frac{12\times5}{2\times15} = 2$$

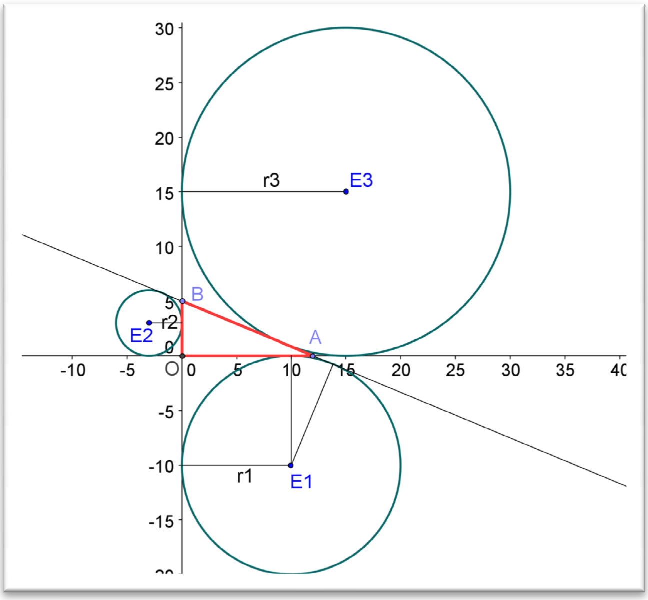
Conclusion : In-centre : E = (r, r) = (2, 2)

In-circle : $(x-2)^2 + (y-2)^2 = 2^2$

Exercise 3:

Prove that the radius of a general ΔABC (not just right) with sides a, b, c and semi-perimeter s is given by : $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$

(5) Ex-center: An ex-circle of the triangle is a circle lying outside the triangle, tangent to one of its sides and tangent to the extensions of the other two.



Every triangle has three distinct ex-circles, each tangent to one of the triangle's sides. The center of an ex-circle is the intersection of the ***internal bisector*** of one angle and the ***external bisectors*** of the other two.

As in the diagram,

Area of ΔOAB

= Area of $\Delta E_1AB + \Delta E_1OB - \Delta E_1OA$

$$\frac{ab}{2} = \frac{cr_1}{2} + \frac{br_1}{2} - \frac{ar_1}{2} = \left(\frac{b+c-a}{2}\right)r_1 = \left(\frac{a+b+c-2a}{2}\right)r_1$$

$$= \left[\left(\frac{a+b+c}{2}\right) - a\right]r_1 = (s-a)r_1$$

$$\therefore \quad r_1 = \frac{ab}{2(s-a)} = \frac{12\times5}{2(15-12)} = 10$$

Similarly, $r_2 = \frac{ab}{2(s-b)} = \frac{12\times5}{2(15-5)} = 3$, $r_3 = \frac{ab}{2(s-c)} = \frac{12\times5}{2(15-13)} = 15$

Conclusion : Ex-centres : $E_1 = (r_1, -r_1) = (10, -10)$

$E_2 = (-r_2, r_2) = (-3,3)$

$E_3 = (r_3, r_3) = (15,15)$

Exercise 4:

(a) Write down the equations of the ex-circles in the above.

(b) Prove that for a general triangle (not just right) the radii of the ex-circles are

$$r_1 = \sqrt{\frac{s(s-b)(s-c)}{s-a}}, \quad r_2 = \sqrt{\frac{s(s-a)(s-c)}{s-b}}, \quad r_3 = \sqrt{\frac{s(s-a)(s-b)}{s-c}}$$