Harder applications of Heron's formula

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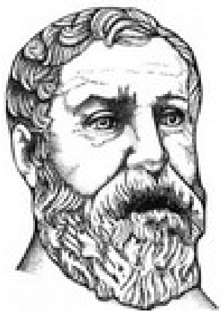
Question 1



You are given a rope of length s, find the largest area of the triangle formed by this rope. What is its area ?

Solution

We are going to prove that the equilateral triangle has the maximum area for any fixed perimeter s.

Let a, b, c be the sides of the triangle .

The perimeter is fixed and $a + b + c = 2s$, so $2s$ is a constant .

By Heron's formula : $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, $s = \frac{a+b+c}{2}$ …. (1)

Now, by Arithmetic mean – Geometric mean inequality (for three variables) ,

$$(s-a)(s-b)(s-c) \le \left[\frac{(s-a)+(s-b)+(s-c)}{3}\right]^3 = \left[\frac{3s-a-b-c}{3}\right]^3 = \left[\frac{3s-2s}{3}\right]^3 = \frac{s^3}{27} \quad …. (2)$$

and the equality occurs when $s - a = s - b = s - c$ (the variables are all equal) , that is, $a = b = c$.

The triangle is therefore equilateral .

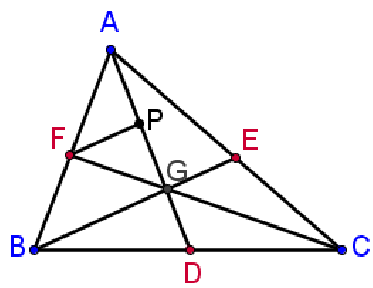
By (2), since the product is always less than a constant $\frac{s^3}{27}$, this constant is the maximum for the product and by Heron's formula, the maximum area of a triangle with fixed perimeter s is

$$\Delta_{max} = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\frac{s(s^3)}{27}} = \frac{s^2}{3\sqrt{3}} = \underline{\underline{\frac{\sqrt{3}s^2}{9}}}$$

Question 2

You are given the medians of the triangle m_a, m_b and m_c. Find the area of ΔABC .

Solution



As in the diagram, D, E, F are mid-points of BC, CA and AB.

G is the centroid . $m_a = AD$, $m_b = BE$ and $m_c = CF$.

Now, $AG = \frac{2}{3}AD$

$\therefore \quad \Delta AGF = \frac{1}{2}\Delta ABG = \frac{1}{6}\Delta ABC$

Let P be the mid-point of AG , we have $PG = \frac{1}{3}m_a$, $FP = \frac{1}{3}m_b$, $FG = \frac{1}{3}m_c$

$$\Delta FPG = \frac{1}{2}\Delta AFG = \frac{1}{12}\Delta ABC$$

By Heron's formula,

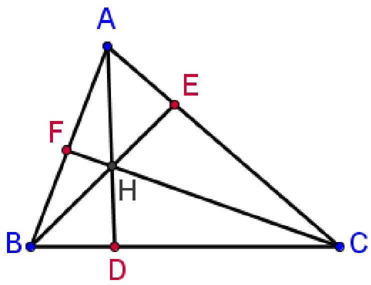
$$\therefore \quad \Delta ABC = 12\Delta FPG = 12 \times \frac{1}{9}\sqrt{m(m-m_a)(m-m_b)(m-m_c)} \quad \text{where} \quad m = \frac{1}{2}(m_a + m_b + m_c)$$

$$= \frac{4}{3}\sqrt{m(m-m_a)(m-m_b)(m-m_c)}$$

Question 3

You are given the altitudes of the triangle h_a, h_b and h_c. Find the area of ΔABC.

Solution



As in the diagram, $AD \perp BC$, $BE \perp CA$, $CF \perp AB$.

G is the orthocentre $h_a = AD$, $h_b = BE$ and $h_c = CF$.

Since $\Delta = \frac{1}{2}h_a a = \frac{1}{2}h_b b = \frac{1}{2}h_c c$, we have :

$$a = \frac{2\Delta}{h_a}, b = \frac{2\Delta}{h_b}, c = \frac{2\Delta}{h_c} \quad \ldots\ldots \quad (3)$$

$$s = \frac{a+b+c}{2} = \Delta\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) \quad \ldots\ldots \quad (4)$$

Put (3), (4) in Heron's formula : $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$\Delta^2 = \left\{\Delta\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\right\}\left\{\Delta\left(-\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\right\}\left\{\Delta\left(\frac{1}{h_a} - \frac{1}{h_b} + \frac{1}{h_c}\right)\right\}\left\{\Delta\left(\frac{1}{h_a} + \frac{1}{h_b} - \frac{1}{h_c}\right)\right\}$$

$$\therefore \quad \Delta = \sqrt{\frac{1}{\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(-\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} - \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} + \frac{1}{h_b} - \frac{1}{h_c}\right)}}$$

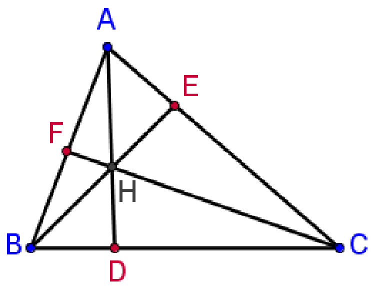
Question 4

You are given the altitudes of the triangle h_b, h_c and side a . Show that the area of ΔABC, Δ,

satisfies the equation : $16\left(\frac{1}{h_b^2} - \frac{1}{h_c^2}\right)^2 \Delta^4 - 8\left[a^2\left(\frac{1}{h_b^2} + \frac{1}{h_c^2}\right) - 2\right]\Delta^2 + a^4 = 0$. If $h_b = 4, h_c = \frac{7\sqrt{2}}{2}$ and

$a = 5$, find Δ.

Solution



As in the diagram, $AD \perp BC$, $BE \perp CA$, $CF \perp AB$.

G is the orthocentre $h_a = AD$, $h_b = BE$ and $h_c = CF$.

We have $b = \frac{2\Delta}{h_b}, c = \frac{2\Delta}{h_c} \quad \ldots\ldots \quad (5)$

Put (5) in Heron's formula : $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$\Delta = \frac{1}{4}\left[a + \left(\frac{1}{h_b} + \frac{1}{h_c}\right)2\Delta\right]^{1/2}\left[-a + \left(\frac{1}{h_b} + \frac{1}{h_c}\right)2\Delta\right]^{1/2}\left[a + \left(\frac{1}{h_b} - \frac{1}{h_c}\right)2\Delta\right]^{1/2}\left[a - \left(\frac{1}{h_b} - \frac{1}{h_c}\right)2\Delta\right]^{1/2}$$

$$16\Delta^2 = \left[\left(\frac{1}{h_b} + \frac{1}{h_c}\right)^2 4\Delta^2 - a^2\right]^{1/2}\left[a^2 - \left(\frac{1}{h_b} - \frac{1}{h_c}\right)^2 4\Delta^2\right]^{1/2}$$

After expansion, we can get an equation : $16\left(\frac{1}{h_b^{\ 2}} - \frac{1}{h_c^{\ 2}}\right)^2 \Delta^4 - 8\left[a^2\left(\frac{1}{h_b^{\ 2}} + \frac{1}{h_c^{\ 2}}\right) - 2\right]\Delta^2 + a^4 = 0$

This is a bi-quadratic equation in Δ , we can solve for unique **positive real root** .