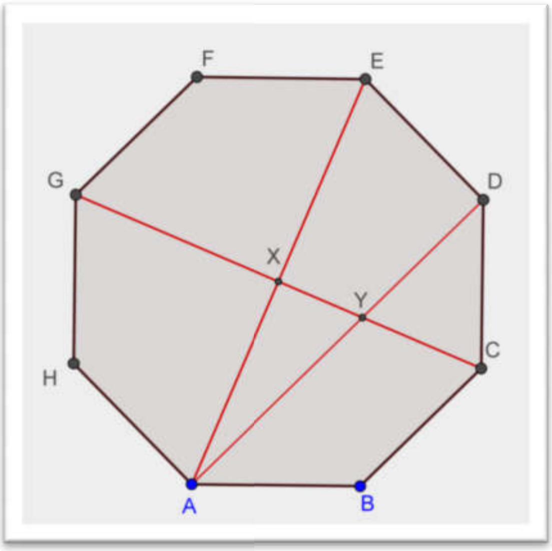
Polygons

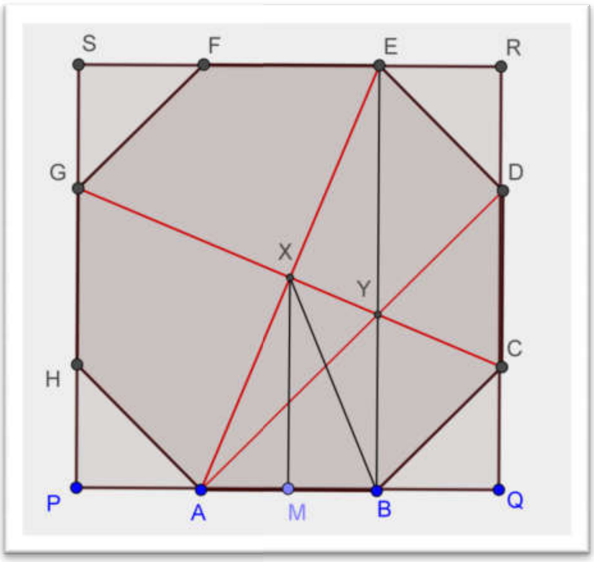
1. (Warm-up on Pythagoras Theorem)
Given a regular octagon ABCDEFGH with sides 2 cm.

(a) If AE intersects CG at X.
Find the length of AX.

(b) If AD cuts CG at Y.
Find the length of XY.



1. **(a)** Draw a square outside touching the sides of the octagon as in the diagram.
It can be seen that X is the centre of both the octagon and the square.



By Pythagoras Theorem on ΔPAH,

$PA = PH = \sqrt{2}$

$PQ = PA + AB + BQ = \sqrt{2} + 2 + \sqrt{2}$

$= 2 + 2\sqrt{2}$

So the side of the square is $2 + 2\sqrt{2}$.

By Pythagoras Theorem on ΔABE,

$$AE = \sqrt{2^2 + 2 + \left(2\sqrt{2}\right)^2} = \sqrt{8\sqrt{2} + 16}$$

$$AX = \frac{1}{2}AE = \frac{1}{2}\sqrt{8\sqrt{2} + 16} = \boldsymbol{\sqrt{2\sqrt{2} + 4}} \approx \mathbf{2.6131259297528} \textbf{ cm}$$

(b) Note that ΔABY is right angled and $\angle BAY = 45°$.

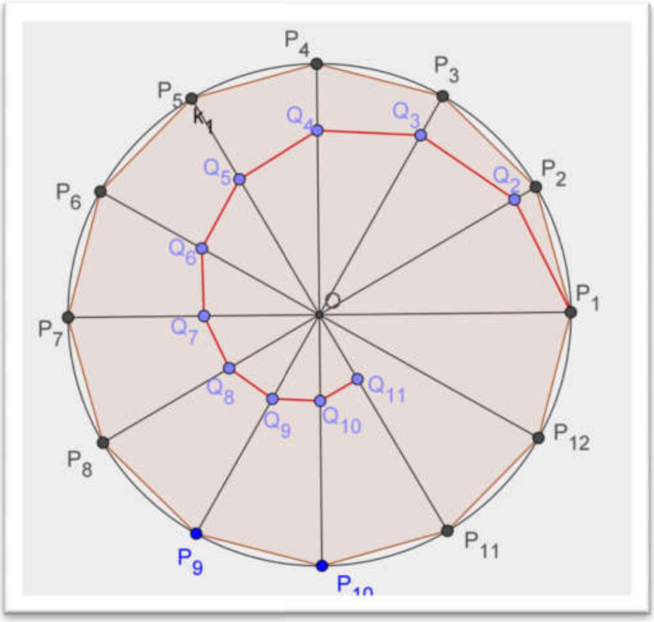
By Pythagoras Theorem, $AY = \sqrt{8}$

Note that AE is perpendicular to CG.

Apply Pythagoras Theorem on ΔAXY .

$$XY = \sqrt{\left(\sqrt{8}\right)^2 - \left(\sqrt{2\sqrt{2} + 4}\right)^2} = \boldsymbol{\sqrt{4 - 2\sqrt{2}}} \approx \mathbf{1.0823922002924} \textbf{ cm}$$

2. A dodecagon of is placed inside a circle of radius 1 cm, and the twelve dividing points are joined to the circle's centre, producing twelve rays. Starting from P_1 a segment is drawn perpendicular to the next ray OP_2 in the anti-clockwise sense; and from the foot of this perpendicular another perpendicular segment is drawn to the next ray, and so on forever. **Taking** $\mathbf{Q_1 = P_1}$.



(a) Find the limit of the sum of the lengths of these segments:

$Q_1Q_2 + Q_2Q_3 + Q_3Q_4 + Q_4Q_5 + \cdots$

$= \sum_{k=1}^{\infty} Q_kQ_{k+1}$

(b) Find the limit of the area of the triangles :

$\Delta OQ_1Q_2 + \Delta OQ_2Q_3 + \Delta OQ_3Q_4 + \cdots$

$= \sum_{k=1}^{\infty} \Delta OQ_kQ_{k+1}.$

(c) (For more able students)

Instead of starting with the circle divided into twelve equal parts, we now to divide it into n equal parts. Let $\angle Q_1OQ_2 = \alpha$.

(i) Find the sum of the lengths: $\sum_{k=1}^{\infty} Q_kQ_{k+1}$

(ii) Find the limit of the area of the triangles : $\sum_{k=1}^{\infty} \Delta OQ_kQ_{k+1}$

2. **(a)** Consider the triangle ΔOQ_kQ_{k+1} .

$\angle OQ_kQ_{k+1} = 60°, \angle Q_kOQ_{k+1} = 30°, \angle Q_kQ_{k+1}O = 90°$

Let $OQ_k = r_k,\ OQ_{k+1} = r_{k+1}$.

Then $r_{k+1} = r_k \cos 30° = \frac{\sqrt{3}}{2} r_k$

$Q_kQ_{k+1} = r_k \sin 30° = \frac{1}{2} r_k$

$Q_{k+1}Q_{k+2} = \frac{1}{2} r_{k+1} = \frac{1}{2}\left(\frac{\sqrt{3}}{2} r_k\right) = Q_kQ_{k+1}\frac{\sqrt{3}}{2}$

Sum of the lengths:

$\sum_{k=1}^{\infty} Q_kQ_{k+1} = Q_1Q_2 + Q_2Q_3 + Q_3Q_4 + Q_4Q_5 + \cdots$

$= \frac{1}{2} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^3 + \cdots$, which is an infinite geometric series

$= \frac{1}{2}\left(\frac{1}{1-\frac{\sqrt{3}}{2}}\right)$, where the common ratio $\frac{\sqrt{3}}{2} < 1$.

$= \frac{1}{2-\sqrt{3}} = \mathbf{2 + \sqrt{3} \approx 3.7320508075689}$ cm

(b) $\mathrm{OQ}_{k+1} = r_{k+1} = \frac{\sqrt{3}}{2} r_k$

The limit of the area of the triangles : $\Delta \mathrm{OQ}_1\mathrm{Q}_2 + \Delta \mathrm{OQ}_2\mathrm{Q}_3 + \Delta \mathrm{OQ}_3\mathrm{Q}_4 + \cdots$

$= \frac{1}{2}\mathrm{OQ}_2 \times \mathrm{Q}_1\mathrm{Q}_2 + \frac{1}{2}\mathrm{OQ}_3 \times \mathrm{Q}_2\mathrm{Q}_3 + \frac{1}{2}\mathrm{OQ}_4 \times \mathrm{Q}_3\mathrm{Q}_4 + \cdots$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2\left[\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right] + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^3\left[\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2\right] + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^4\left[\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^3\right] + \cdots$

$= \left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}\right)^3 + \left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}\right)^5 + \left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}\right)^7 + \cdots$

$= \left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{1-\left(\frac{\sqrt{3}}{2}\right)^2}\right)$, where the common ratio $\left(\frac{\sqrt{3}}{2}\right)^2 < 1$

$= \boldsymbol{\frac{\sqrt{3}}{2} \approx 0.8660254037844}$ cm

(c) **(i)** Consider the triangle $\Delta \mathrm{OQ}_k\mathrm{Q}_{k+1}$.

$\angle \mathrm{Q}_k\mathrm{OQ}_{k+1} = \alpha,\ \angle \mathrm{Q}_k\mathrm{Q}_{k+1}\mathrm{O} = 90°$

Let $\mathrm{OQ}_k = r_k,\ \mathrm{OQ}_{k+1} = r_{k+1}$.

Then $r_{k+1} = r_k \cos\alpha$

$\mathrm{Q}_k\mathrm{Q}_{k+1} = r_k \sin\alpha$

$\mathrm{Q}_{k+1}\mathrm{Q}_{k+2} = r_{k+1}\sin\alpha = r_k \sin\alpha\cos\alpha = \mathrm{Q}_k\mathrm{Q}_{k+1}\cos\alpha$

Sum of the lengths:

$\sum_{k=1}^{\infty} \mathrm{Q}_k\mathrm{Q}_{k+1} = \mathrm{Q}_1\mathrm{Q}_2 + \mathrm{Q}_2\mathrm{Q}_3 + \mathrm{Q}_3\mathrm{Q}_4 + \mathrm{Q}_4\mathrm{Q}_5 + \cdots$

$= (\sin\alpha) + (\sin\alpha)(\cos\alpha) + (\sin\alpha)(\cos\alpha)^2 + (\sin\alpha)(\cos\alpha)^3 + \cdots$

$= (\sin\alpha)\frac{1}{1-\cos\alpha} = \boldsymbol{\frac{\sin\alpha}{1-\cos\alpha}}$

(ii) Note that : $\mathrm{OQ}_{k+1} = r_{k+1} = r_k\cos\alpha$

The limit of the area of the triangles :

$\Delta \mathrm{OQ}_1\mathrm{Q}_2 + \Delta \mathrm{OQ}_2\mathrm{Q}_3 + \Delta \mathrm{OQ}_3\mathrm{Q}_4 + \cdots$

$= \frac{1}{2}\mathrm{OQ}_2 \times \mathrm{Q}_1\mathrm{Q}_2 + \frac{1}{2}\mathrm{OQ}_3 \times \mathrm{Q}_2\mathrm{Q}_3 + \frac{1}{2}\mathrm{OQ}_4 \times \mathrm{Q}_3\mathrm{Q}_4 + \cdots$

$= \frac{1}{2}\cos\alpha\sin\alpha + \frac{1}{2}(\cos\alpha)^2[\sin\alpha\cos\alpha] + \frac{1}{2}(\cos\alpha)^3[\sin\alpha(\cos\alpha)^2]$

$+\frac{1}{2}(\cos\alpha)^4[\sin\alpha(\cos\alpha)^3] + \cdots$

$= \frac{1}{2}\sin\alpha\cos\alpha + \frac{1}{2}\sin\alpha(\cos\alpha)^3 + \frac{1}{2}\sin\alpha(\cos\alpha)^5 + \frac{1}{2}\sin\alpha(\cos\alpha)^7 + \cdots$

$= \frac{1}{2}\sin\alpha\cos\alpha\frac{1}{1-(\cos\alpha)^2} = \frac{\sin\alpha\cos\alpha}{2(\sin\alpha)^2} = \frac{\cos\alpha}{2\sin\alpha} = \boldsymbol{\frac{1}{2\tan\alpha}}$

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