Young's Inequality

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Question 1

Let f be a real-valued function which is continuously differentiable and strictly increasing on the interval $\mathbf{I} = [0, \infty)$. Suppose $f(0) = 0$. Let $a \in \mathbf{I}$ and $b \in f(\mathbf{I})$.

(a) For any $a \in \mathbf{I}$, define $g(t) = bt - \int_0^t f(x)dx$.

Prove that g attains its maximum value at $f^{-1}(b)$.

(b) **(i)** Prove that $\int_0^{f^{-1}(b)} xf'(x)dx = g(f^{-1}(b))$.

(ii) By changing a variable, prove that $\int_0^{f^{-1}(b)} xf'(x)dx = \int_0^b f^{-1}(x)dx$.

(c) Use **(a)** and **(b)** to show that $\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \ge ab$. Draw a diagram to show the geometric meaning of this inequality if the integrals are interpreted as areas.

(d) Use **(c)** to show the Young's inequality : $\frac{a^p}{p} + \frac{b^q}{q} \ge ab$, where $p > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$

Solution

(a) Since f is strictly increasing, $f'(t) > 0$.

$g(t) = bt - \int_0^t f(x)dx$

$\Rightarrow \quad g'(t) = b - \frac{d}{dt}\int_0^t f(x)dx = b - f(x)$, by the Fundamental Theorem of Integral Calculus.

$\therefore \quad g'(t) = 0$ iff $t = f^{-1}(b)$.

Since $g''(t) = -f'(t) < 0$, since $f'(t) > 0$.

$\therefore \quad g$ attains its maximum value at $f^{-1}(b)$.

(b) **(i)** Using integration by parts, $\int_0^{f^{-1}(b)} xf'(x)dx = xf(x)\Big|_0^{f^{-1}(b)} - \int_0^{f^{-1}(b)} f(x)dx$

$$= bf^{-1}(b) - \int_0^{f^{-1}(b)} f(x)dx = \left[bt - \int_0^t f(x)dx\right]_{t=f^{-1}(b)} = g(t)\Big|_{t=f^{-1}(b)} = g(f^{-1}(b))$$

(ii) Put $y = f(x)$, or $x = f^{-1}(y)$. Then $f'(x)\,dx = dy$

When $x = f^{-1}(b)$, $y = b$. When $x = 0$, $y = 0$ (since $f(0) = 0$)

$\therefore \quad \int_0^{f^{-1}(b)} xf'(x)dx = \int_0^b f^{-1}(y)dy = \int_0^b f^{-1}(x)dx$, since y is a dummy variable.

(c) From **(a)**, $g(f^{-1}(b)) \ge g(t)$, where $t \in \mathbf{I}$. In particular, $g(f^{-1}(b)) \ge g(a)$ …. (1)

From the definition of $g(t)$, $g(a) = b \times a - \int_0^a f(x)dx$ …. (2)

From **(b) (i)** and **(ii)** , $\int_0^b f^{-1}(x)dx = g\left(f^{-1}(b)\right)$ …. (3)

$\therefore \quad \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx = [ab - g(a)] + g\left(f^{-1}(b)\right)$, by (2) and (3)

$\geq ab$, by (1) .

$\int_0^a f(x)dx = \text{Area III}$

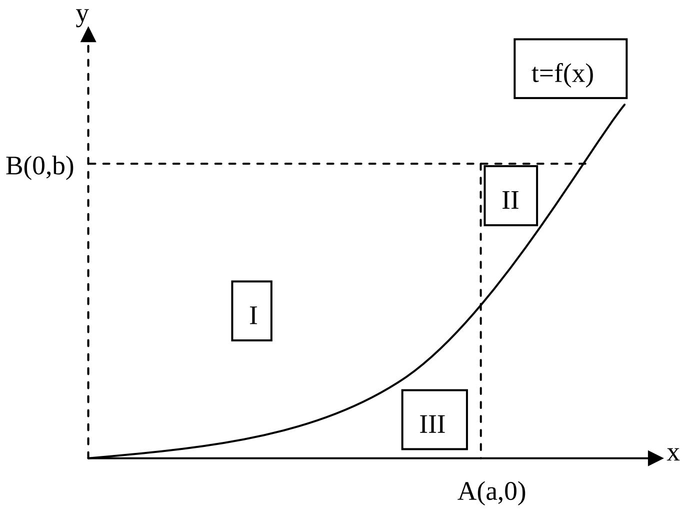
$\int_0^b f^{-1}(x)dx = \text{Area I} + \text{Area II}$.

$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx$

$= \text{Area I} + \text{Area II} + \text{Area III}$

$\geq \text{Area I} + \text{Area III}$

$= ab$



(d) From **(c)** , $\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab$.

Put $y = f(x) = x^{p-1}$, $p > 2$ in the above inequality .

Since $y' = (p-1)x^{p-2} > 0$ for $x \in \mathbf{I}$, we have

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx = \int_0^a x^{p-1}dx + \int_0^b x^{\frac{1}{p-1}}dx = \int_0^a x^{p-1}dx + \int_0^b x^{q-1}dx \quad \text{, since } \frac{1}{p} + \frac{1}{q} = 1$$

$$= \left.\frac{x^p}{p}\right|_0^a + \left.\frac{x^q}{q}\right|_0^b = \frac{a^p}{p} + \frac{b^q}{q}$$

$\therefore \quad \frac{a^p}{p} + \frac{b^q}{q} \geq ab$, by **(c)** .

Question 2 [Young's inequality ⇒ Generalized A.M. ≥ G.M.]

(a) Given that $\lambda\alpha + (1-\lambda)\beta \geq \alpha^{\lambda}\beta^{1-\lambda}$, for $0 < \lambda < 1$, $\alpha, \beta \geq 0$.

Prove that if $\frac{1}{p} + \frac{1}{q} = 1$, then $\frac{a^p}{p} + \frac{b^q}{q} \geq ab$. (Young's inequality)

(b) If $p, q, x, y > 0$ such that $p + q = 1$, prove that $px + qy \geq x^p y^q$.

(c) If $\alpha_1, \alpha_2, ..., \alpha_{m+1}$; $x_1, x_2, ..., x_{m+1} > 0$, such that $\alpha_1 + \alpha_2 + ... + \alpha_{m+1} = 1$, by letting $p = \alpha_1 + \alpha_2 + ... + \alpha_m$ and $x = x_1 + x_2 + ... + x_m$, or otherwise, prove that :

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_{m+1} x_{m+1} \geq \left(\frac{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m}{\alpha_1 + \alpha_2 + ... + \alpha_m} \right)^p x_{m+1}{}^{\alpha_{m+1}} .$$

(d) Prove by induction : If $\alpha_1, \alpha_2, ..., \alpha_m$; $x_1, x_2, ..., x_m > 0$, such that $\alpha_1 + \alpha_2 + ... + \alpha_m = 1$,

then $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m \ge x_1^{\alpha_1} x_2^{\alpha_2} .... x_m^{\alpha_m}$.

(e) If $p_1, p_2, ..., p_m$; $x_1, x_2, ..., x_m > 0$, prove that

$$\left(x_1^{p_1} x_2^{p_2} .... x_m^{p_m} \right)^{\frac{1}{p_1 + p_2 + ... + p_m}} \le \frac{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m}{p_1 + p_2 + ... + p_m}$$

Solution

(a) Let $\lambda = \frac{1}{p}$, then $1 - \lambda = \frac{1}{q}$. Also let $a = \alpha^p$, $b = \beta^q$. $\therefore$ $\frac{a^p}{p} + \frac{b^q}{q} \ge ab$

(b) Let $a = x^{1/p}$, $b = y^{1/q}$ $\therefore$ $\frac{x}{p} + \frac{y}{q} \ge x^{1/p} + y^{1/q}$.

Further replace $\frac{1}{p}$ by p and $\frac{1}{q}$ by q . $\therefore$ $p + q = 1$ and $px + qy \ge x^p y^q$.

(c) $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_{m+1} x_{m+1} = x + \alpha_{m+1} x_{m+1} = p\left(\frac{x}{p}\right) + \alpha_{m+1} x_{m+1}$

$\ge \left(\frac{x}{p}\right)^p x_{m+1}^{\alpha_{m+1}}$, by **(b)** , since $p + \alpha_{m+1} = 1$

$= \left(\frac{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m}{\alpha_1 + \alpha_2 + ... + \alpha_m}\right)^p x_{m+1}^{\alpha_{m+1}}$

(d) The assertion is trivial for $m = 1$.

Assume the assertion is true for some integer $m \ge 1$.

$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_{m+1} x_{m+1} \ge \left(\frac{\alpha_1}{p} x_1 + \frac{\alpha_2}{p} x_2 + ... + \frac{\alpha_m}{p} x_m\right)^p x_{m+1}^{\alpha_{m+1}}$, by **(c)** , since $\frac{\alpha_1}{p} + ... + \frac{\alpha_m}{p} = 1$

$\ge \left(x_1^{\alpha_1 / p} x_2^{\alpha_2 / p} .... x_m^{\alpha_m / p} \right)^p x_{m+1}^{\alpha_{m+1}}$, by inductive hypothesis.

$= x_1^{\alpha_1} x_2^{\alpha_2} .... x_m^{\alpha_m} x_{m+1}^{\alpha_{m+1}}$

$\therefore$ The assertion is also true for the integer $m + 1$. Result follows by induction.

(e) Replace $\alpha_i = \frac{p_i}{p_1 + p_2 + ... + p_m}$ where $i = 1, 2, \dots, m$ in **(d)** . Then

$\alpha_1 + \alpha_2 + ... + \alpha_m = 1$ and $\left(x_1^{p_1} x_2^{p_2} .... x_m^{p_m} \right)^{\frac{1}{p_1 + p_2 + ... + p_m}} \le \frac{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m}{p_1 + p_2 + ... + p_m}$