Mathematical Induction

Proofs by Induction

1. Prove the following by induction:

(a) $1+2+3+\ldots+n=\frac{1}{2}n(n+1)$

(b) $1^2+2^2+3^2+\ldots+n^2=\frac{1}{6}n(n+1)(2n+1)$

(c) $x^{2n}-y^{2n}$ is divisible by x + y for any integers x, y and positive integer n.

(d) $1^3+2^3+3^3+\ldots+n^3=\left[\frac{1}{2}n(n+1)\right]^2$

(e) $2\times 1+3\times 2+4\times 2^2+5\times 2^3+\ldots$ to n terms $= n2^n$

(f) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}$

(g) $2^{4n}-1$ is divisible by 15.

(h) There are $\frac{1}{2}n(n-3)$ diagonals in a convex n-sided polygon $(n\geq 3)$.

(i) $m, n\geq 2$, $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_m\in \mathbf{R}$,

$$(a_1+a_2+\ldots+a_n)(b_1+b_2+\ldots+b_m)=\sum_{j=1}^{m}\left(\sum_{i=1}^{n}a_ib_j\right)=\sum_{i=1}^{n}\left(\sum_{j=1}^{m}a_ib_j\right)$$

(When m = n = 2, what formula does this reduce to ?)

(j) $3^{n-2}\geq n^5$ for $n\geq 20$.

(k) $2^n>n^2$ for $n\geq 5$.

(l) $\sum_{n=1}^{m}\frac{n(n+1)(n+2)\ldots(n+p-1)}{p!}=\frac{m(m+1)(m+2)\ldots(m+p)}{(p+1)!}$

(m) $C^n_r=\frac{n!}{r!(n-r)!}$ is always an integer.

(n) $2\times 4^{2n+1}+3^{3n+1}$ is divisible by 11.

(o) $4^{2n+1}+3^{n+2}$ is divisible by 13.

(p) $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}<1$

(q) The number of pairs of non-negative integers (x, y) satisfying $x+2y=n$ is $\frac{1}{2}(n+1)+\frac{1}{4}\left[1+(-1)^n\right]$.

2. **(a)** Prove that 2^{n+1} is a factor of $\left(\sqrt{3}+1\right)^{2n+1}-\left(\sqrt{3}-1\right)^{2n+1}$ $\forall\, n\in \mathbf{N}\cup\{0\}$.

(b) Prove that $\left(3+\sqrt{5}\right)^n+\left(3-\sqrt{5}\right)^n$ is divisible by 2^n for all positive integers n.

3. Two persons play the following game. Take two collections of marbles of the same quantity. Each player can take out any number of marbles from any collection each time. They are not allowed to take from both collections at one time. The one who picks the last marble wins. Show that the one who starts later wins.

4. **(a)** Show that $\forall n \in \mathbf{N}$, $\left(\sqrt{m^2+1}-m\right)^n = a_n\sqrt{m^2+1}-b_n$, where $a_n, b_n \in \mathbf{Z}$, $m \in N$.

(b) Show that for any $n \in \mathbf{N}$, we can find a natural number N with $\left(\sqrt{m^2+1}-m\right)^n = \sqrt{N+1}-\sqrt{N}$.

5. Prove by induction,

$$x_1 + x_2(1+x_1) + x_3(1+x_1)(1+x_2) + \ldots + [x_n(1+x_1)\ldots(1+x_{n-1})] = (1+x_1)(1+x_2)\ldots(1+x_n) - 1.$$

Deduce a well-known formula by putting $x_1 = x_2 = \ldots = x_n = x$.

6. Show that the sum of the first n odd positive integers is a perfect square.

The odd positive integers are arranged in blocks with n integers in the nth block, so that

$a_1 = 1$, $a_2 = 3 + 5$, $a_3 = 7 + 9 + 11$, etc.

Find an expression for a_n and deduce that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

Show that: $\sum_{r=1}^{n} r^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$

by considering the situation in which the nth block has n^2 integers.

7. **(i)** Let f(x) be a convex function defined on [a, b], i.e. $f(x_1)+f(x_2) \le 2f\left(\frac{x_1+x_2}{2}\right)$ for all $x_1, x_2 \in [a, b]$.

For each positive integer n, consider the statement:

I(n) : If $x_i \in [a, b]$, $i = 1, 2, \ldots, n$, then $f(x_1)+\ldots+f(x_n) \le nf\left(\frac{x_1+\ldots+x_n}{n}\right)$.

(a) Prove by induction that $I(2^k)$ is true for every positive integer k.

(b) Prove that if I(n) $(n \ge 2)$ is true, then I(n-1) is true.

(c) Prove that I(n) is true for every positive integer n.

(ii) Prove that $f(x) = \sin x$ is convex on $[0, \pi]$ and hence that

$$\frac{1}{n}(\sin\theta_1 + \sin\theta_2 + \ldots + \sin\theta_n) \le \sin\frac{\theta_1+\theta_2+\ldots+\theta_n}{n} \qquad \text{for } 0 \le \theta_i \le \pi.$$

8. Let us have a sequence of numbers (Fibonacci's series): 0, 1, 1, 2, 3, 5, 8, 13, 21, ….

This sequence is determined by the following conditions: $u_{n+1} = u_n + u_{n-1}$ $(n \ge 1)$ and $u_0 = 0, u_1 = 1$.

(i) Show that there exist the following relations:

(a) $u_{n+2} = u_0 + u_1 + u_2 + \ldots + u_n + 1$

(b) $u_{2n+2} = u_1 + u_3 + u_5 + \ldots + u_{2n+1}$

(c) $u_{2n+1} = 1 + u_2 + u_4 + \ldots + u_{2n}$

(d) $-u_{2n-1} + 1 = u_1 - u_2 + u_3 - \ldots + u_{2n-1} - u_{2n}$

(e) $u_{2n-2} + 1 = u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1}$

(f) $u_n u_{n+1} = u_1^2 + u_2^2 + \ldots + u_n^2$

(g) $u_{2n}^2 = u_1u_2 + u_2u_3 + \ldots + u_{2n-1}u_{2n}$

(h) $u_{n+1}u_{n+2} - u_n u_{n+3} = (-1)^n$

(i) $u_n^2 - u_{n+1}u_{n-1} = (-1)^{n+1}$

(j) $u_n^4 - u_{n-2}u_{n-1}u_{n+1}u_{n+2} = 1$

(ii) Compute the sum $\frac{1}{1\cdot 2}+\frac{2}{1\cdot 3}+...+\frac{u_{n+2}}{u_{n+1}u_{n+3}}$.

(iii) Prove the relations:

(a) $u_{n+p-1} = u_{n-1}u_{p-1} + u_n u_p$

(b) $u_{2n-1} = u_n^2 + u_{n-1}^2$

(c) $u_{2n-1} = u_n u_{n+1} - u_{n-2}u_{n-1}$

(iv) Prove that: $u_n^3 + u_{n+1}^3 - u_{n-1}^3 = u_{3n}$.

(v) Prove that: $u_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} C_{n-k-1}^k$, where $\left[\frac{n-1}{2}\right]$ is the largest integer smaller than or equal to $\frac{n-1}{2}$.

9. Prove the identity:

$$\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+...+\left(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}\right)=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+\frac{1}{2n-1}-\frac{1}{2n}\right).$$

10. Prove the following identities:

(i) $\sum_{x=1}^{x=n} x(x+1)...(x+q)=\frac{1}{q+2}n(n+1)...(n+q+1)$

(ii) $\sum_{x=1}^{x=n} \frac{1}{x(x+1)...(x+q)}=\frac{1}{q}\left\{\frac{1}{q!}-\frac{1}{n(n+1)...(n+q)}\right\}$.

11. Prove that: $\sum_{k=1}^{k=n} \frac{(1-a^n)(1-a^{n-1})...(1-a^{n-k+1})}{1-a^k}=n$.

12. Let $F_n(z)=\frac{q}{1-q}(1-z)+\frac{q^2}{1-q^2}(1-z)(1-qz)+...+\frac{q^n}{1-q^n}(1-z)(1-qz)...(1-q^{n-1}z)$,

Prove the identity: $1 + F_n(z) - F_n(qz) = (1 - qz)(1 - q^2z) \dots (1 - q^nz)$.

13. Prove the identity:

$$\frac{b+c+d+...+k+\ell}{a(a+b+c+...+k+\ell)}=\frac{b}{a(a+b)}+\frac{c}{(a+b)(a+b+c)}+\frac{d}{(a+b+c)(a+b+c+d)}+...+\frac{1}{(a+b+...+k)(a+b+...+k+\ell)}$$

14. Prove the validity of the identity:

$$1+\frac{1}{a}+\frac{a+1}{ab}+\frac{(a+1)(b+1)}{abc}+...+\frac{(a+1)(b+1)...(s+1)(k+1)}{abc...sk\ell}=\frac{(a+1)(b+1)...(s+1)(k+1)(\ell+1)}{abc...sk\ell}$$

15. Check the identity: $(1+x)(1+x^2)(1+x^4)...(1+x^{2^{n-1}})=1+x+x^2+x^3+...+x^{(2^n-1)}$.

16. Prove the identity: $\frac{x}{1-x^2}+\frac{x^2}{1-x^4}+\frac{x^4}{1-x^8}+...+\frac{x^{2^{n-1}}}{1-x^{2^n}}=\frac{1}{1-x}\frac{x-x^{2^n}}{1-x^{2^n}}$.

17. Prove the identity: $\frac{n}{2n+1}+\left[\frac{1}{2^3-2}+...+\frac{1}{(2n)^3-2n}\right]=\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}$

18. Let the pairs of numbers: (a, b), (a_1, b_1), (a_2, b_2), … be obtained according to the following law:

$$a_1 = \frac{a+b}{2},\ b_1 = \frac{a_1+b}{2},\ a_2 = \frac{a_1+b_1}{2},\ b_2 = \frac{a_2+b_1}{2}, \dots .$$

Prove that:

$$a_n = a + \frac{2}{3}(b-a)\left(1 - \frac{1}{4^n}\right), \qquad b_n = a + \frac{2}{3}(b-a)\left(1 + \frac{1}{2\times 4^n}\right).$$

19. Let the real numbers p_n and q_n be defined by:

$p_n = \frac{1}{2}(p_{n-1} + q_{n-1})$ and $p_n q_n = k^2$, where k is a positive constant and p_0, q_0 are fixed real numbers.

If $p_0 > k > q_0$, show that $p_{n-1} > p_n > k > q_n > q_{n-1} > 0$

and if $p_n = k(1 + 2\lambda_n)$, show that $\lambda_n < {\lambda_0}^{2n}$.

20. If $u_n = \dfrac{x(x+1)(x+2)\dots(x+n-1)}{n!}$, show that

(a) $\sum_{n=1}^{p} u_n(x) = u_p(x+1) - 1$

(b) $0 < u_n\left(\frac{1}{2}\right) < \frac{1}{\sqrt{2n+1}}$.

Hence show that $1 > \frac{1}{2} + \frac{1}{2\cdot 4} + \frac{1\cdot 3}{2\cdot 4\cdot 6} + \dots + \frac{1\cdot 3\cdot 5\cdots(2p-3)}{2\cdot 4\cdot 6\cdots(2p)} > 1 - \frac{1}{\sqrt{2p+1}}$.

21. Let $\ell_n = \sqrt{a + \ell_{n-1}}$, $\ell_1 = \sqrt{a}$. Show that $\ell_n < \sqrt{a} + 1$ and $\ell_{n-1} < \ell_n$.

22. **(a)** Prove that $\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$,

hence evaluate: $1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \cdots + n\cdot 1$.

(b) By mathematical induction, show that: $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \le 2\sqrt{n}$.

Hence deduce that: $\left(\frac{n}{4}\right)^n < n! < \left[\frac{1}{6} n(n+1)(2n+1)\right]^{\frac{n}{2}}$.

23. Let $a_1, a_2, \dots$ be an arithmetic sequence of positive real numbers. Prove that:

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \dots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}$$