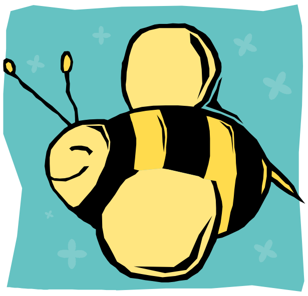
The mathematics of honeycomb

By So Hon Cheong(6E)

Nature is amazing. Things are always arranged in elegant patterns. Actually, we can often use mathematical concepts to analyse these patterns. Many animals and even plants in nature are mathematicians. In this article, we will concentrate on bees-their honeycombs are great works of maths.



Everyone knows that honeycomb consists of hexagons. But why?

The most important reason is that hexagon is one of the few types of regular polygons that can cover up a space completely without coinciding. If a regular polygon is to fulfil the above requirement, the sum of certain number of its interior angles must be equal to 2π.

Interior angle in a n-sided regular polygon = $\frac{\pi(n-2)}{n}$

Thus, $\frac{\pi(n-2)}{n}(m) = 2\pi$ where m is a positive integer

$$\frac{2n}{n-2} = m$$

We can divide the problem into several cases for discussion:

(A) If n is an odd number, then n-2 is also an odd number

(1) When the denominator n-2=1, $\frac{2n}{n-2}$ is always an integer, at this time n=3

(2) When the denominator n-2≥ 3, since n-2 is odd, if 2n is divisible by n-2, n must be divisible by n-2; but n=(n-2)+2, 2 is not divisible by (n-2), so n is NOT a multiple of n-2.

Thus, n can only be 3 if it is odd.

(B) If n is an even number, n-2 is also even

Let n=2k, where k is a positive integer

$$\frac{2n}{n-2} = \frac{2(2k)}{2k-2} = \frac{2k}{k-1}$$

(1) When k=2, the denominator k-1=1, $\frac{2k}{k-1}$ is always an integer, this time

n=4

(2) when k=3, the denominator k-2=2, $\frac{2k}{k-1}$ is always an integer, this time

n=6

(3) When k>3 , since 2k=2(k-1)+2, 2(k-1) is divisible by k-1 but 2 is not divisible by k-1, $\frac{2k}{k-1}$ is never an integer.

Thus, n can only be 4 or 6 when n is even.

In conclusion, amongst regular polygons, only triangles (n=3), squares (n=4) and hexagons (n=6) can fill a space completely without coinciding.

Then the next question comes up: why don't the bees choose triangles or squares?

Let us consider the perimeter of a triangle, a square and a regular hexagon of the same area.

(1) If the length of one side of the triangle=n,

its area $= \frac{1}{2} n^2 \sin \frac{\pi}{3}$

Perimeter = 3n

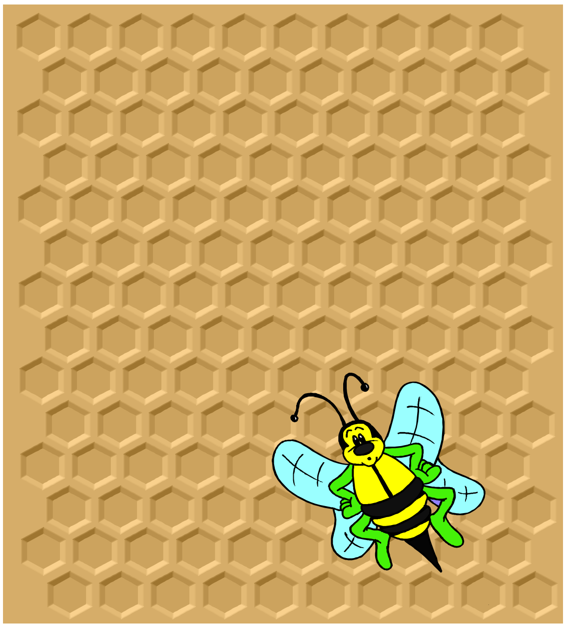
(2) Let a side of the square=n

Area of square = area of triangle

$$s^2 = \frac{1}{2} n^2 \sin \frac{\pi}{3}$$

$$s = \sqrt{\frac{1}{2} \sin \frac{\pi}{3}}\, n$$

$$\text{Perimeter} = 4\sqrt{\frac{1}{2} \sin \frac{\pi}{3}}\, n$$



(3) Let a side of the hexagon be t

Area of hexagon = area of triangle

$$\frac{1}{2} t^2 \sin \frac{\pi}{3} \times 6 = \frac{1}{2} n^2 \sin \frac{\pi}{3}$$

$$t = \frac{n}{\sqrt{6}}$$

$$\text{Perimeter} = \frac{6}{\sqrt{6}}\, n$$

$3^4 = 81$

$$\left(4\sqrt{\frac{1}{2}\sin\frac{\pi}{3}}\right)^4 \quad = 48$$

$$\left(\frac{6}{\sqrt{6}}\right)^4 \qquad = 36$$

$$\left(\frac{6}{\sqrt{6}}\right) < \left(4\sqrt{\frac{1}{2}\sin\frac{\pi}{3}}\right) < 3$$

Perimeter of hexagon < perimeter of square < perimeter of triangle

Since the perimeter of hexagon is the smallest for a fixed area, bees can use the least amount of wax to build the cells.

Nature is wonderful and full of mysteries. There are indeed plenty of miracles in nature that are worth exploring.